Variational iteration solving method of a sea-air oscillator model for the ENSO*

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Abstract A time delay equation for sea-air oscillator model is studied. The aim is to create an asymtotic solving method of nonlinear equation for the ENSO model. And based on a class of oscillators of ENSO model, employing the variational iteration method, the approximate solution of corresponding problem is obtained. It is proven from the results that the method of variational iteration method can be used for analyzing the sea surface temperature anomaly in the equatorial eastern Pacific of the atmosphere-ocean oscillation for ENSO model.

Keywords: **Nonlinear problem, time delay, El Niño-Southern oscillator model; variational iteration.**

During the last decade, many studies focused on the interannual climate variability associated with the El the Niño-Southern Oscillation (ENSO). Progress has been made in understanding and simulating the coupled tropical ocean-atmosphere system by using models with varying complexity[1]. A mechanism for the oscillatory nature of ENSO was originally proposed by McCreary[2]. Suarez and Schopf[3] introduced the conceptual delayed oscillator as a candidate mechanism for ENSO, by considering the effects of equatorially trapped oceanic waves propagating in a closed basin through a delay term. The conceptual delayed oscillator model is represented by a single ordinary differential delay equation with both positive and negative feedbacks. The positive feedback is represented by the local ocean-atmosphere coupling in the equatorial eastern Pacific. The delayed negative feedback is represented by free Rossby waves generated in the eastern Pacific coupling region, which propagate to and reflect from the western boundary, returning as Kelvin waves to reverse the anomalies in the eastern Pacific coupling region.

The ENSO is a natural phenomenon involved in the tropical Pacific ocean-atmosphere interactions. Phenomenon of the ENSO is a very attractive subject of study in the international academic circles. Many phenomena are considered, such as slowdown of the meridional overturning circulation by McPhaden and Zhang[4], interdecadal climate fluctuations by Gu and Philander[5], interaction change of the structure of the ENSO mode by An and Wang[6], decadal variations in the subtropical cells by Nonaka, Xie and McCreary[7], a unified oscillator model for the ENSO by Wang[8]. Mo et al.[9—11] have studied a class of nonlinear differential equations[9] and nonlinear problems in atmospheric physics, ocean weather, dynamic system[10,11] by using the fix point theorem and the perturbation theorem.

Recently, many scholars have investigated the approximate theory of the nonlinear problems. Approximate methods have been developed and refined, including the method of averaging, boundary layer method, methods of matched asymptotic expansion and multiple scales, such as Hwangm[12] and Khasminskii and Yin[13], have done a great deal of work. He[14—16] also provided the following categories of asymptotic: variational approaches, parameter-expanding method, parameterized perturbation method, homotopy perturbation method and iteration perturbation method. In this paper we consider a sea-air oscillator model by using a simple and valid variational iteration method[14].

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1 Model of equatorial Pacific SST

The delayed oscillator model does not consider the effect of the western Pacific on ENSO. It is assumed that winds in the western Pacific do not affect the sea surface temperature (SST) anomalies in the eastern Pacific and the atmosphere is in a steady state or the Niño-4 (over 5°S—5°N, 160°E—150°W) zonal wind stress anomalies are linearly proportional to the Niño-3 (over 5°S—5°N, 150°S—90°W) SST anomalies. The delay oscillator model to be investigated is given as follows:[8]

$$\frac{dT}{dt} = \frac{ad}{R}T - \frac{bd}{R}T(t-\eta) - \varepsilon T^3, \qquad (1)$$

where T is the Niño-3 SST anomaly, η is a time for waves to travel to the western boundary and return to the eastern Pacific, a is a coefficient representing the positive feedback, b is a coefficient representing the negative feedback due to waves reflection at the western boundary, d is a positive coefficient that relates the Niño-3 SST anomalies to the Niño-4 zonal wind stress anomalies, R is a damping coefficient and ε is a cubic damping coefficient.

2 Variational iteration solution of SST

According to the variational iteration method proposed by He[14], a correction functional can be constructed as follows.

Introducing a functional $F(T,\lambda)$:

$$F(T,\lambda) = T - \int_0^t \lambda\left(\frac{dT}{d\tau} - \frac{ad}{R}T - \frac{bd}{R}\bar{T}(t-\eta) + \varepsilon\bar{T}^3\right)d\tau, \qquad (2)$$

where $\bar{T}$ is a restricted variable[14] of T, while λ is the Lagrange multiplicator.

Compute the variations δF of functional (2):

$$\delta F = \delta T - (\lambda\delta T)\mid_{\tau=t} + \int_0^t\left(\lambda'\delta T + \frac{ad}{R}\lambda\delta T\right)d\tau.$$

Let $\delta F = 0$. Thus we have

$$\frac{d\lambda}{d\tau} + \frac{ad}{R}\lambda = 0, \qquad (3)$$

$$\lambda(t) = 1. \qquad (4)$$

From (4), we have

$$\lambda = \exp\left(-\frac{ad}{R}(\tau - t)\right). \qquad (5)$$

From (2) and (5), we construct the following variational iteration:

$$T_{n+1} = T_n - \int_0^t \exp\left(-\frac{ad}{R}(\tau - t)\right) \cdot \left(\frac{dT_n(\tau)}{d\tau} - \frac{ad}{R}T_n(\tau) - \frac{bd}{R}T_n(\tau-\eta) + \varepsilon T_n^3(\tau)\right)d\tau. \qquad (6)$$

Obviously, if we yield a convergent sequence of functions $\{T_n(t)\}$ from (6), then $T(t) = \lim_{n\to\infty} T_n(t)$ is a solution of (1).

Now we first select an initial approximate solution $T_0(t)$ that it is a solution for the linear equation of (1):

$$\frac{dT_0}{dt} = \frac{ad}{R}T_0. \qquad (7)$$

From (7), we have

$$T_0(t) = C\exp\left(\frac{ad}{R}t\right), \qquad (8)$$

where C is an arbitrary constant.

Because the nonlinear term of Eq. (1) is $\bar{R} = -\varepsilon T^3$, we have $\lim_{\|T\|\to 0}\frac{\|\bar{R}\|}{\|T\|} = 0$, where $\|\bar{R}\| = \varepsilon T^3$, $\|T\| = T$. Then the zero solution of the nonlinear equation (1) is the same stable behaviour as the zero solution of the linear equation (7). Then the zero solution of the equation (1) is instable too.

Substituting initial approximate solution (8) into (6), we can obtain the first order approximate solution of the equation (1):

$$T_1(t) = \left[1 + \frac{bdt}{R}\exp\left(-\frac{ad}{R}\eta\right) + \frac{\varepsilon RC^2}{2ad} \cdot \left(\exp\left(\frac{2ad}{R}t\right) - 1\right)\right]C\exp\left(\frac{ad}{R}t\right), \qquad (9)$$

where C is an arbitrary constant.

Using the same method, we can obtain a higher order approximate solution of Eq. (1).

3 Accuracy of approximate solution

In order to show the accuracy of the above result, now we make a comparison as the following special case.

We give a set of parameters as follows[13]:

$$a = 1.5\times 10^2 ℃\ m^2N^{-1}yr^{-1},$$
$$b = 0 ℃\ m^2N^{-1}yr^{-1},$$
$$d = 3.6\times 10^{-2} ℃\ m^{-2}Nyr^{-1},$$
$$\eta = 0\ yr^{-1},\quad \varepsilon = 0.1 ℃\ yr^{-1},\quad R = 2.0\ yr^{-1}.$$

Now we establish a comparison of the values of between T_{num} for the numerical calculation and T_{app}

of the approximate expansion (8) in the case of values ε (10^{-2}℃$^{-2}$ yr^{-1}) and t (yr). We obtain the following results (see Fig.1 and Table 1):

Table 1. Comparison of the values of between T_{num} and T_{app}

	$t=0.00$	$t=0.05$	$t=0.10$	$t=0.15$	$t=0.20$	$t=0.25$	$t=0.30$	$t=0.35$	$t=0.40$	$t=0.45$	$t=0.50$	$t=0.55$
T_{num}	1.00	1.14	1.31	1.50	1.71	1.96	2.24	2.56	2.92	3.34	3.81	4.34
T_{app}	1.00	1.14	1.31	1.50	1.72	1.97	2.26	2.57	2.97	3.40	3.91	4.49

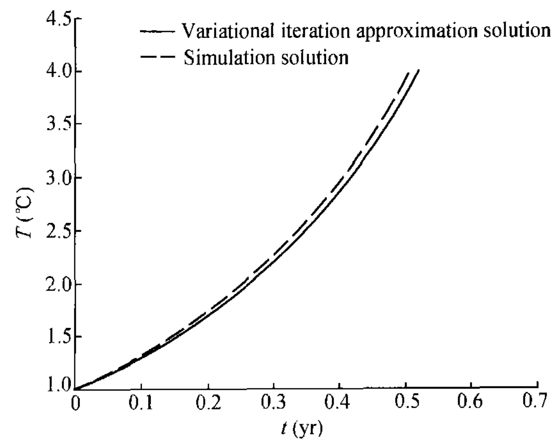

Fig. 1. Comparison between simulation solution T_{num} and variational iteration approximate solution T_{app}.

Comparing the solution of the numerical calculation, we know that approximate solution (9) obtained by using the variational iteration method possesses finer approximate degree.

Then from expression (9), we can analyze the SST anomaly in the equatorial eastern Pacific of the sea-air oscillation for ENSO model (1).

4 Conclusions

(i) From the comparison of accuracy, we can conclude that the obtained T_n from the variational iteration (6) is a good approximate expansion of the solution for the nonlinear time delay model (1).

(ii) The ENSO is a complicated natural phenomenon. Hence we need to reduce basic models for the sea-air oscillator and solve them by using the approximate method. The method of variational iteration is a simple and valid method.

(iii) The method of variational iteration is an approximate method, which differs from the general numerical method. The expansions of solution through the method of variational iteration can be kept in the analytic operation. Thus, from Eq. (9), we can further study the fixed quality and quantitative behavior of the temperature anomaly in the equatorial eastern Pacific.

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